AD-A046 376 WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
POINTWISE BIVARIATIONAL BOUNDS ON SOLUTIONS OF FREDHOLM INTEGRA--ETC(U)
JUL 77 P D ROBINSON DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1767 NL

|OF|

AD
A046376

END
DATE
FILMED
12-77
DDC

(11)

ADA046376

(9) MRC Technical Summary Report, #1767

(6) POINTWISE BIVARIATIONAL BOUNDS ON SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS.

(10) Peter D. Robinson

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

(11) July 1977

(12) 17p.

Received June 1, 1977

(15) DAAG29-75-C-0024

(14) MRC-TSR-1767

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

1473

221 200

LB

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

POINTWISE BIVARIATIONAL BOUNDS ON SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

Peter D. Robinson†

Technical Summary Report # 1767

July 1977

ABSTRACT

Pointwise bounds of a bivariational nature are derived on the solution of a standard Fredholm integral equation of the second kind, with a symmetric kernel. The bounding functionals involve two trial vectors, one approximating the solution and the other approximating the reciprocal kernel. Even with the latter taken as the null vector, there is significant improvement over a previous approach. It is shown how suitable choices of trial vector lead to expressions for bounds on Neumann, Padé and Fredholm approximations.

AMS(MOS) Subject Classification - 45L05, 45G20

Key Words - Integral equation, Symmetric kernel, Pointwise bivariational bounds, Hilbert space, Neumann, Padé, Fredholm.

Work Unit Number 1 - Applied Analysis

EXPLANATION

It is not often possible to solve exactly an integral equation like

$$\phi(x) + \lambda \int_a^b k(x,y)\, \phi(y)\, dy = f(x), \qquad a \le x \le b ,$$

for the unknown function $\phi(x)$ in terms of the known functions $f(x)$ and $k(x,y)$ (the 'kernel' function, symmetric in x and y). So we derive analytical expressions for two other functions $\phi_-(x)$ and $\phi_+(x)$ with the property

$$\phi_-(x') \le \phi(x') \le \phi_+(x'), \qquad a \le x' \le b ,$$

providing upper and lower bounds on $\phi(x')$ at all points of the interval. The differences $\phi_+(x') - \phi(x')$ and $\phi(x') - \phi_-(x')$ have the same kind of size as $\Delta \quad \delta(x')$, where

† On leave from Bradford University, England.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and by a grant from the University of Wisconsin-Madison Graduate School.

$$\Delta = \{\int_a^b [\Phi(x) - \phi(x)]^2 \, dx\}^{1/2}$$

and

$$\delta(x') = \{\int_a^b [\Psi(x,x') - \psi(x,x')]^2 \, dx\}^{1/2} .$$

Here $\Phi(x)$ is any approximation to $\phi(x)$, and $\Psi(x,x')$ is any approximation to the solution $\psi(x,x')$ of the equation

$$\psi(x,x') + \lambda \int_a^b k(x,y)\,\psi(y,x')\,dy = k(x,x') .$$

There are several ways of finding reasonable approximate solutions to integral equations, and these can be used to make Δ and $\delta(x')$ very small. Thus tight "pointwise bounds" ϕ_- and ϕ_+ can be obtained on ϕ. Some bounds are evaluated numerically for a simple integral equation, and found to be very tight indeed. The method can also be used indirectly to find pointwise bounds on some of the standard analytical approximations to ϕ.

POINTWISE BIVARIATIONAL BOUNDS ON SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

Peter D. Robinson†

1. Introduction.

In this paper, pointwise bounds of a bivariational nature are derived on the solutions of a standard Fredholm integral equation of the second kind, with a symmetric kernel. The method depends on improved bivariational bounds associated with linear equations in a Hilbert space, and exploits the similarity between the integral equation itself and the equation specifying the reciprocal kernel. The bounding functionals involve two trial vectors, one approximating the solution of the integral equation and the other approximating the reciprocal kernel. Even when the latter is taken to be the null vector, the bounds refine those which can be obtained from an approach of Lonseth [1]; a simple numerical illustration is given. It is also shown how suitable choices of trial vector lead to expressions for bounds on approximate solutions of Neumann, Padé and Fredholm type.

2. Hilbert Space Formulation.

The pointwise bounds are obtained as instances of complementary (upper and lower) bivarational bounds on the linear product $\langle g,\phi\rangle$ associated with a pair of equations

$$A\phi = f\ , \qquad \phi, f \in \mathcal{H} \tag{2.1}$$

$$A\psi = g\ , \qquad \psi, g \in \mathcal{H} \tag{2.2}$$

in a Hilbert space $\mathcal{H}$, A being a self-adjoint operator on $\mathcal{H}$. We assume that A is strictly positive and bounded below away from zero, so that for some positive number β

$$\langle\Phi, A\Phi\rangle \geq \beta\langle\Phi,\Phi\rangle \quad \text{for all } \Phi \in \mathcal{H}\ . \tag{2.3}$$

Then the inverse operator A^{-1} exists with domain the whole of $\mathcal{H}$. Further, we assume that A is bounded above, so that for some positive number α

$$\alpha\langle\Phi,\Phi\rangle \geq \langle\Phi, A\Phi\rangle \text{ for all } \Phi \in \mathcal{H}, \quad \alpha > \beta > 0\ . \tag{2.4}$$

(The case $\alpha \to \infty$ is admissible, but means that A is <u>not</u> bounded above; then A is merely defined <u>in</u> $\mathcal{H}$ and wherever necessary vectors must be assumed to belong to the domain of A).

The Fredholm integral equation with real symmetric kernel

$$\phi(x) + \lambda\int_a^b k(x,y)\phi(y)dy = f(x), \qquad a \leq x \leq b\ , \tag{2.5}$$

† On leave from Bradford University, England

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and by a grant from the University of Wisconsin-Madison Graduate School.

is identified with (2.1). If $\psi(x,y)$ is the reciprocal (or resolvent) kernel, so that the relation

$$\phi(x) + \lambda \int_a^b \psi(x,y)\, f(y)dy = f(x), \qquad a \leq x \leq b , \tag{2.6}$$

specifies the solution $\phi(x)$ of (2.5), then it is a standard result that

$$\psi(x,x') + \lambda \int_a^b k(x,y)\, \psi(y,x')dy = k(x,x'), \qquad a \leq x \leq b, \quad a \leq x' \leq b . \tag{2.7}$$

This equation (2.7) is identified with (2.2). We take for $\mathcal{H}$ the real space of square-integrable functions with inner product

$$<h_1,h_2> = \int_a^b h_1(x)\, h_2(x)dx, \quad \text{for all } h_1,h_2 \in \mathcal{H} , \tag{2.8}$$

although it is possible to adapt the theory for complex spaces.

The symmetric kernel $k(x,y)$ is assumed to be of Hilbert-Schmidt type, and describes a compact, self-adjoint operator K in $\mathcal{H}$, so that

$$A = I + \lambda K . \tag{2.9}$$

The reciprocal kernel $\psi(x,y)$ is also symmetric, and describes the reciprocal operator Γ given by

$$\Gamma = K(I + \lambda K)^{-1} = KA^{-1} . \tag{2.10}$$

Both kernels, regarded as functions of x, can play the role of vectors in $\mathcal{H}$. Thus, taking

$$g(x) = k(x,x') , \tag{2.11}$$

it follows from equation (2.5) and the symmetry of $k(x,x')$ that

$$<g,\phi> = \int_a^b k(x,x')\, \phi(x)dx = \lambda^{-1}\{f(x') - \phi(x')\} . \tag{2.12}$$

Accordingly complementary bounds on $<g,\phi>$ lead to complementary bounds on $\phi(x')$, for any x' in $a \leq x' \leq b$.

The strict positivity requirement (2.3) places a restriction on the values of the parameter λ for which the results will be valid. Since K is compact and self-adjoint, it follows that for some non-negative numbers M and L,

$$M < \phi,\phi > \;\geq\; < \phi,K\phi > \;\geq\; -L< \phi,\phi > \quad \text{for all } \phi \in \mathcal{H} . \tag{2.13}$$

In theory, the numbers $+M$ and $-L$ are respectively the positive and negative eigenvalues of K of greatest magnitude, but in practice they can be taken as the best available upper and lower bounds to these eigenvalues. Then we assume that

$$L^{-1} > \lambda > -M^{-1} \tag{2.16}$$

and take

$$\left.\begin{aligned} \alpha &= 1 + \lambda M, \quad \beta = 1 - \lambda L \quad \text{if} \quad \lambda > 0 \,, \\ \alpha &= 1 + \lambda L, \quad \beta = 1 + \lambda M \quad \text{if} \quad \lambda < 0 \,. \end{aligned}\right\} \tag{2.15}$$

The bivariational bounds derived in §3 below involve the positive constants ξ and η defined by

$$\xi = \tfrac{1}{2}(\beta^{-1}+\alpha^{-1}), \quad \eta = \tfrac{1}{2}(\beta^{-1}-\alpha^{-1}) \,. \tag{2.16}$$

For either positive or negative λ, it follows from (2.15) that

$$\xi = \frac{1 + \frac{1}{2}\lambda(M-L)}{(1+\lambda M)(1-\lambda L)}, \quad \eta = \frac{\frac{1}{2}|\lambda|(M+L)}{(1+\lambda M)(1-\lambda L)} \,. \tag{2.17}$$

3. Complementary Bivariational Bounds

Associated with an equation

$$A\theta = h, \quad \theta, h \in \mathcal{H} \,, \tag{3.1}$$

whose solution θ is unknown, are the complementary variational bounds on $\langle h,\theta\rangle$:

$$G_\alpha(\Theta;h) \le \langle h,\theta\rangle \le G_\beta(\Theta,h), \quad \text{for all } \Theta \in \mathcal{H} \,, \tag{3.2}$$

where

$$G_\beta(\Theta;h) = -\langle \Theta, A\Theta\rangle + 2\langle h,\Theta\rangle + \alpha^{-1}\|A\Theta - h\|^2 \tag{3.3}$$

and

$$G_\beta(\Theta;h) = -\langle \Theta, A\Theta\rangle + 2\langle h,\Theta\rangle + \beta^{-1}\|A\Theta - h\|^2 \,. \tag{3.4}$$

The norm here is the usual Hilbert space norm, $\|\phi\|^2 = \langle \phi,\phi\rangle$. The variational bounding properties of G_α and G_β follow at once from the identities

$$G_\alpha(\Theta;h) = \langle h,\theta\rangle - \alpha^{-1}\langle A^{1/2}\delta\theta, (\alpha-A)A^{1/2}\delta\theta\rangle \,, \tag{3.5}$$

and

$$G_\beta(\theta;h) = \langle h,\theta\rangle + \beta^{-1}\langle A^{1/2}\,\delta\theta,(A-\beta)A^{1/2}\delta\theta\rangle \ , \tag{3.6}$$

in terms of the difference-vector $\delta\theta = \theta - \theta$, together with (2.3) and (2.4).

Consider now the pair of equations

$$A(s\phi + t\psi) = sf + tg \ , \tag{3.7}$$

$$A(s\phi - t\psi) = sf - tg \ , \tag{3.8}$$

obtained from (2.1) and (2.2), s and t being scalar multipliers. If we subtract the two inner products like $\langle h, \theta\rangle$ which can be bounded as in (3.2) we obtain:

$$\begin{aligned}\langle(sf+tg), (s\phi+t\psi)\rangle - \langle(sf-tg), (s\phi-t\psi)\rangle &= 2st\langle g, \phi\rangle + 2st\langle f,\psi\rangle \\ &= 4st\langle g,\phi\rangle \ .\end{aligned} \tag{3.9}$$

(The last step follows since $\langle f,\psi\rangle = \langle A\,\phi,\psi\rangle = \langle\phi,A\psi\rangle = \langle\phi,g\rangle = \langle g,\phi\rangle$) . Thus, from (3.2) and (3.9), we have for all $\Phi,\Psi \in \mathcal{H}$:

$$\begin{aligned}&G_\alpha(sf + tg;\ s\Phi + t\Psi) - G_\beta(sf - tg;\ s\Phi - t\Psi) \le \\ &\le 4st\langle g, \phi\rangle \le G_\beta(sf+tg;s\Phi+t\Psi) - G_\alpha(sf-tg;s\Phi-t\Psi) \ .\end{aligned} \tag{3.10}$$

The left-hand and right-hand members of (3.10) reduce to

$$\begin{aligned}&4st\{-\langle\Psi,A\Phi\rangle + \langle\Psi,f\rangle + \langle g,\Phi\rangle\} + \\ &4\xi st\langle A\Phi-f,A\Psi-g\rangle \mp 2\eta\{s^2\,\|A\Phi-f\|^2 + t^2\,\|A\Psi-g\|^2\} \ .\end{aligned} \tag{3.11}$$

Dividing (3.10) through by $4|st|$, and choosing the optimal ratio

$$|s|\,\|A\Phi-f\| = |t|\,\|A\Psi-g\| \ , \tag{3.12}$$

we obtain the final result

$$J(\Psi,\Phi) + \xi S(\Psi,\Phi) - \eta C(\Psi,\Phi) \le \langle g,\phi\rangle \le J(\Psi,\Phi) + \xi S(\Psi,\Phi) + \eta C(\Psi,\Phi) \tag{3.13}$$

where

$$J(\Psi,\Phi) = -\langle\Psi,A\,\Phi\rangle + \langle\Psi,f\rangle + \langle g,\Phi\rangle = \langle g,\phi\rangle - \langle\delta\psi,A\phi\rangle \ , \tag{3.14}$$

$$S(\Psi,\Phi) = \langle A\Phi-f,\ A\Psi-g\rangle = \langle A\,\delta\phi,A\delta\psi\rangle \ , \tag{3.15}$$

$$C(\Psi,\Phi) = \|\Lambda\Psi-g\| \, \|\Lambda\Phi-f\| = \|\Lambda\delta\psi\| \, \|\Lambda\delta\phi\| \tag{3.16}$$

and

$$\delta\phi = \Phi - \phi \in \mathcal{H}, \quad \delta\psi = \Psi - \psi \in \mathcal{H} . \tag{3.17}$$

The expression of the functionals J, S and C in terms of the difference-vectors (3.17) makes clear the bivariational nature of the bounds in (3.13). They are tighter than others previously obtained [2] by exploiting the identity

$$<(sf \pm s^{-1}g), (s\phi \pm s^{-1}\psi)> - s^2<f,\phi> - s^{-2}<g,\psi> = \pm 2<g,\phi> \tag{3.18}$$

rather than (3.9); see also [3].

4. Application to the Integral Equations

When applied to the pair of integral equations (2.5) and (2.7), the functionals in (3.13)-(3.16) take the forms

$$\begin{aligned} J(x') &= \lambda^{-1}\{f(x') - \phi(x')\} - <\delta\psi,(1 + \lambda K)\delta\phi> \\ &= -\int_a^b \Psi(x,x') \{(1+\lambda K) \Phi(x) - f(x)\}dx + \int_a^b k(x,x') \Phi(x)dx , \end{aligned} \tag{4.1}$$

$$\begin{aligned} S(x') &= <(1+\lambda K)\delta\phi, (1+\lambda K)\delta\psi> \\ &= \int_a^b \{(1+\lambda K) \Phi(x) - f(x)\}\{(1+\lambda K) \Psi(x,x') - k(x,x')\}dx , \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} C(x') &= \|(1+\lambda K)\delta\phi\| \, \|(1+\lambda K)\delta\psi\|_x \\ &= [\int_a^b \{(1+\lambda K)\Phi(x) - f(x)\}^2 dx]^{1/2} [\int_a^b \{(1+\lambda K)\Psi(x,x') - k(x,x')\}^2 dx]^{1/2} . \end{aligned} \tag{4.3}$$

Whenever there is ambiguity, the operator K is understood to act on the first argument of the subsequent function, so that

$$K\Psi(x,x') = \int_a^b k(x,y) \Psi(y,x')dy . \tag{4.4}$$

Substituting from (2.12) into (3.13), we obtain the pointwise bounds

$$\phi_+(x') \geq (x') \geq \phi_-(x'), \quad a \leq x' \leq b , \tag{4.5}$$

where

$$\phi_+(x') = f(x') - \lambda\{J(x') + \xi S(x')\} + |\lambda|\eta\, C(x') \qquad (4.6)$$

and

$$\phi_-(x') = f(x') - \lambda\{J(x') + \xi S(x')\} - |\lambda|\eta\, C(x') \ . \qquad (4.7)$$

The functions $\Phi(x)$ and $\Psi(x,x')$, which determine $J(x')$, $S(x')$ and $C(x')$, are intended to be approximations to the exact solutions $\phi(x)$ and $\psi(x,x')$ of equations (2.5) and (2.7). The upper bound $\phi_+(x')$ can be minimized with respect to any disposable parameters in Φ and Ψ (for example multipliers of some set of basis vectors), and likewise $\phi_-(x')$ can be maximized. Thus we have a method for determining accurate bounds on $\phi(x')$; an example is presented in §5.

A simpler alternative procedure is to forego individual optimization of ϕ_+ and ϕ_-, and write (4.5) in the form

$$0 \le |\phi(x') - [f(x') - \lambda\{J(x') + \xi S(x')\}]| \le |\lambda|\eta\, C(x') \ . \qquad (4.8)$$

This gives a pointwise measure of the accuracy of the function

$$\tilde{\phi}(x') = f(x') - \lambda\{J(x') + \xi S(x')\} \qquad (4.9)$$

as an approximation to $\phi(x')$. Disposable parameters might then be chosen to minimize the "least-squares" function $C(x')$. Bounds on various analytical approximations to ϕ, such as the Neumann, Padé and Fredholm approximations, can also be obtained from (4.8). By suitable choice of the trial vectors Φ and Ψ it is possible to coax $f(x') - \lambda J(x')$ into the form of one or other of these approximations. [See §6 below].

5. Simple Trial Functions

Sensible results are obtained from the foregoing theory even with the uninformed choice of trial functions which are identically zero. When both $\Phi(x) = 0$ and $\Psi(x,x') = 0$, (4.8) gives rise to the pointwise bounds

$$0 \le |\phi(x') - [f(x') - \lambda\xi K f(x')]| \le |\lambda|\eta \|f\| \, \|k(x,x')\|_x . \qquad (5.1)$$

With zero $\Psi(x,x')$, but arbitrary $\Phi(x)$, (4.6) and (4.7) become

$$\phi_\pm(x') = f(x') - \lambda K\Phi(x') + \lambda\xi K\{\Phi(x') + \lambda K\Phi(x') - f(x')\} \pm |\lambda|\eta \|(1+\lambda K)\Phi - f\| \, \|k(x,x')\|_x \ . \qquad (5.2)$$

Lonseth [1] has in effect pointed out that in any case $\phi(x')$ lies between

$$f(x') - \lambda K\phi(x') \pm |\lambda| \, \|\delta\phi\| \, \|k(x,x')\|_x \; ; \tag{5.3}$$

bounds on the Hilbert space norm $\|\delta\phi\|$ then lead to pointwise bounds on $\phi(x')$. Using (2.3), these would be

$$f(x') - \lambda K\phi(x') \pm |\lambda|\beta^{-1} \|(1+\lambda K)\phi - f\| \, \|k(x,x')\|_x \tag{5.4}$$

with β given by (2.15), which is evidently a coarser result than (5.2) since $\beta^{-1} > 2\eta$ from (2.16).

If $\phi(x')$ is taken as $cf(x')$ in (5.2), and the factor c is optimized to give the best upper and lower bounds, these turn out to be

$$\tilde{\phi}_{\pm}(x') = \phi_0(x') + u_1 p(x') \pm u_2\{r(x')^2 - p(x')^2\}^{1/2} \; , \tag{5.5}$$

where

$$\left.\begin{aligned}
\phi_0(x') &= f(x') - \lambda\xi K f(x') \; , \\
p(x') &= \xi(\lambda K + \lambda^2 K^2) \, f(x') - \lambda K f(x') \; , \\
r(x') &= \eta\|(1+\lambda K)f\| \, \|\lambda k\|_x \; , \\
u_1 &= \frac{\langle f,(1+\lambda K)f\rangle}{\|(1+\lambda K)f\|^2} \; , \\
u_2^2 &= \frac{\|f\|^2}{\|(1+\lambda K)f\|^2} - \frac{\langle f,(1+\lambda K)f\rangle^2}{\|(1+\lambda K)f\|^4} \; .
\end{aligned}\right\} \tag{5.6}$$

But in order to expoit fully the bivariational nature of the pointwise bounds in (4.5)-(4.7), the crude choice $\Psi(x,x') = 0$ must be relaxed. If we choose

$$\phi(x') = cf(x'), \quad \Psi(x,x') = dk(x,x') \tag{5.7}$$

in (4.1)-(4.3), and simultaneously determine c and d in each case to optimize the bounds (4.6) and (4.7), these become

$$\phi_{\pm}^{\dagger} = \phi_0 + \frac{p^2}{q} - \frac{(p-qu_1)(p-qv_1)t^2}{q(t^2-q^2)} \tag{5.8}$$

$$\pm \frac{t}{t^2-q^2}\{(t^2-q^2)u_2^2 - (p-qu_1)^2\}^{1/2}\{(t^2-q^2)v_2^2 - (p-qv_1)^2\}^{1/2} \; .$$

Here, in addition to (5.6), we have used the notation

$$\left.\begin{aligned} q(x') &= \xi(\lambda K + 2\lambda^2K^2 + \lambda^3K^3)f - (\lambda K + \lambda^2K^2)\, f(x')\ , \\ t(x') &= \eta\|(1+\lambda K)f\|\ \|\lambda k + \lambda^2 k_2\|_x\ , \\ v_1(x') &= \frac{\langle \lambda k\ ,\lambda k + \lambda^2 k_2\rangle_x}{\|\lambda k + \lambda^2 k_2\|_x^2}\ , \\ v_2^2(x') &= \frac{\|\lambda k\|_x^2}{\|\lambda k + \lambda^2 k_2\|_x^2} - \frac{\langle \lambda k\ ,\lambda k + \lambda^2 k_2\rangle_x^2}{\|\lambda k + \lambda^2 k_2\|_x^4}\ , \end{aligned}\right\} \tag{5.9}$$

wherein $k_2(x,x')$ is the second iterated kernel.

Lonseth [1] considered the example

$$\left.\begin{aligned} &\phi(x) + \int_0^1 k(x,y)\,\phi(y)dy = x^2, \qquad 0 \le x \le 1\ , \\ &k(x,y) = x(1-y) \ \text{ if } \ x \le y, \quad k(x,y) = y(1-x) \ \text{ if } \ x \ge y \end{aligned}\right\}\ , \tag{5.10}$$

for which

$$\left.\begin{aligned} &f(x) = x^2,\ \lambda = 1,\ \ M = \pi^{-2},\ L = 0,\ \ \xi = \frac{\pi^2+1/2}{\pi^2+1}\ ,\ \eta = \frac{1/2}{\pi^2+1}\ , \\ &Kf(x) = \frac{x-x^4}{12}\ ,\ K^2f(x) = \frac{4x-5x^3+x^6}{360}\ ,\ \|k(x,x')\|_x = \frac{x'(1-x')}{\sqrt{3}}\ . \end{aligned}\right\} \tag{5.11}$$

The crude bounds in (5.1) give

$$0 \le \left|\phi(x') - [x'^2 - \frac{\xi(x'-x'^4)}{12}]\right| \le (15)^{-1/2}\ \eta x'(1-x'),\quad 0 \le x' \le 1\ . \tag{5.12}$$

At $x' = 1/2$, the least accurate value, this shows that

$$0.21225 \le \ \phi(1/2) \le 0.21819\ . \tag{5.13}$$

The bounds (5.5), with Φ proportional to f and Ψ zero, give

$$0.21704 \le \ \phi(1/2) \le 0.21723\ , \tag{5.14}$$

whereas following Lonseth's approach via (5.4) one obtains only

$$0.21292 \le \ \phi(1/2) \le 0.21740 \tag{5.15}$$

with the same type of Φ . Finally, with Φ proportional to f and Ψ proportional to k, the more sophisticated bounds (5.8) yield the accurate result

$$0.217047 \le \phi(1/2) \le 0.217048 . \tag{5.16}$$

The integral equation (5.5) can be solved exactly if it is converted into a differential equation, and in fact

$$\phi(1/2) = \frac{5}{2}\,\mathrm{sech}(1/2) - 2 = 0.2170472 . \tag{5.17}$$

6. Bounds on Analytical Approximations

6.1. The Neumann approximation

Let us make the choice of trial vectors

$$\Phi(x) = \Phi_n(x), \quad \Psi(x,x') = \Psi_m(x,x'), \quad n,m = 0,1,2... \tag{6.1}$$

in the functionals (4.1)-(4.3), where

$$\left.\begin{aligned} &\Phi_0(x) = 0 , \\ &\Phi_n(x) = \{1 - \lambda K + \lambda^2 K^2 + \ldots + (-\lambda K)^{n-1}\} f(x), \quad n \ge 1 , \end{aligned}\right\} \tag{6.2}$$

and

$$\left.\begin{aligned} \Psi_0(x,x') &= 0 , \\ \Psi_m(x,x') &= \{1 - \lambda K + \lambda^2 K^2 + \ldots + (-\lambda K)^{m-1}\} k(x,x'), \quad m \ge 1 , \\ &= k(x,x') - \lambda k_2(x,x') + \lambda^2 k_3(x,x') \ldots + (-\lambda)^{m-1} k_m(x,x') , \end{aligned}\right\} \tag{6.3}$$

the functions $k_2 \ldots k_m$ being the iterated kernels $(k_1 = k)$. This choice leads to the expressions

$$\left.\begin{aligned} &J(x') = 0, \quad m + n = 0 , \\ &J(x') = \{1 - \lambda K + \ldots + (-\lambda K)^{m+n-1}\} Kf(x'), \quad m + n \ge 1 , \end{aligned}\right\} \tag{6.4}$$

$$S(x') = (-\lambda K)^{m+n} Kf(x') \tag{6.5}$$

and

$$C(x') = |\lambda|^{m+n} \, \|K^n f\| \, \| k_{m+1}(x,x') \|_x . \tag{6.6}$$

From (4.8), (4.9) and (2.17), these results indicate that the Neumann-type approximate solution

$$\tilde{\phi}(x') = \{1 - \lambda K + \ldots + (-\lambda K)^{m+n} + \xi(-\lambda K)^{m+n+1}\} f(x'), \quad a \le x' \le b , \tag{6.7}$$

is in error by not more than

$$\pm \; |\lambda|^{m+n+2} \; \frac{\frac{1}{2}(M+L)}{(1+\lambda M)(1-\lambda L)} \; \| K^n f \| \; \| k_{m+1}(x,x') \|_x \; . \tag{6.8}$$

The relation (2.6) connecting $\phi(x)$ and $\psi(x,y)$ indicates that $n = m+1$ is a sensible condition to impose, but the maximum error in (6.8) holds good for arbitrary n and m.

6.2. The [N/N] Padé approximant

Consider trial vectors of the form

$$\Phi(x) = \sum_{n=0}^{N-1} a_n K^n f(x) \; , \tag{6.9}$$

$$\Psi(x,x') = \sum_{n=0}^{N-1} b_n K^n k(x,x') = \sum_{n=0}^{N-1} b_n k_{n+1}(x,x') \; , \tag{6.10}$$

where the a_n and b_n are disposable constants. The functional $J(x')$ in (4.1) is itself a bivariational approximation to $\lambda^{-1}\{f(x') - \phi(x')\}$, and so it is not unreasonable to choose the constants a_n and b_n to make J stationary with respect to variations in them. If this is done, the function $\{f(x') - \lambda J(x')\}$ can be identified as the [N/N] Padé approximant to $\phi(x')$ in powers of λ, constructed directly from the Neumann Series. Chisholm [4] has suggested this sequence of approximations to ϕ as an alternative to the Fredholm ones, and has proved a convergence result for them. The theory of the present paper gives bounds on them, via (4.8), verifying their accuracy to order λ^{2N+1}.

If we use the notation

$$f_n(x) = K^n f(x) \tag{6.11}$$

and denote by Δ_N the array

$$\left.\begin{array}{llll} (f_1 + \lambda f_2) & (f_2 + \lambda f_3) & \cdots & (f_N + \lambda f_{N+1}) \\ (f_2 + \lambda f_3) & (f_3 + \lambda f_4) & \cdots & (f_{N+1} + \lambda f_{N+2}) \\ \cdots & \cdots & \cdots & \cdots \\ (f_N + \lambda f_{N+1}) & (f_{N+1} + \lambda f_{N+2}) & \cdots & (f_{2N-1} + \lambda f_{2N}) \end{array}\right\} , \tag{6.12}$$

then the functionals $J(x')$, $S(x')$ and $C(x')$ can be written as

$$J(x') = - \begin{vmatrix} 0 & \cdots & f_N(x') \\ f_1(x') & & \\ \vdots & \Delta_N(x') & \\ f_N(x') & & \end{vmatrix} \div |\Delta_N(x')| \tag{6.13}$$

and

$$C(x') = \| \Omega_N f \| \, \| \Omega_N k(x,x') \|_x , \quad S(x') = \Omega_N^2 \, Kf(x') \quad , \tag{6.14}$$

where the operator Ω_N is specified by

$$\Omega_N = \lambda^N \begin{vmatrix} 1 & f_1(x') & \cdots & f_N(x') \\ K & f_2(x') & \cdots & f_{N+1}(x') \\ \vdots & \vdots & & \vdots \\ K^N & f_{N+1}(x') & \cdots & f_{2N}(x') \end{vmatrix} \div |\Delta_N(x')| \ . \tag{6.15}$$

(cf. Barnsley [5]). We note that when $k(x,x')$ is a degenerate kernel of order N, the rows of the numerator-determinant in (6.15) are linearly dependent, and so Ω_N is then the null-operator. Accordingly the $[N/N]$ Padé approximant to ϕ is exact [4].

Bounds on other types of Padé approximant can be obtained by modifying the powers of K in (6.9) and (6.10) (cf. 6).

6.3. The Fredholm approximation

One choice of trial vectors which leads to bounds on the n^{th} Fredholm approximation is

$$\left.\begin{aligned} \Phi(x) &= \left[\frac{E_0+\lambda E_1+\ldots+\lambda^{n-1}E_{n-1}}{d_0+\lambda d_1+\ldots+\lambda^n d_n}\right] f(x), \qquad n = 1,\ 2,\ \ldots \\ \Psi(x) &= 0 \ , \end{aligned}\right\} \tag{6.16}$$

where

$$\left.\begin{aligned} E_0 &= I, \quad E_m = d_m I - KE_{m-1}, & m &= 1,\ 2,\ \ldots \\ d_0 &= 1, \quad md_m = \sum_{r=1}^{m} t_r \, d_{m-r} (-1)^{r+1}, & m &= 1,\ 2,\ \ldots\ , \end{aligned}\right\} \tag{6.17}$$

and t_r is the r^{th} trace of the kernel $k(x,x')$. The operators E_m are polynomials in K, and KE_m has as kernel the classical Fredholm minor; the numbers d_m are the classical Fredholm determinants [7,8].

With the choice (6.16), $f(x') - \lambda J(x')$ is actually the n^{th} Fredholm approximation to $\phi(x')$, and we obtain the expressions

$$J(x') = K\Phi(x') ,$$

$$C(x') = \frac{|\lambda|^n}{|d_0+\lambda d_1+\ldots+\lambda^n d_n|} \|(d_n - KE_{n-1})f\| \, \|k(x,x')\|_x \tag{6.18}$$

and

$$S(x') = \frac{\lambda^n KE_n f(x')}{d_0+\lambda d_1+\ldots+\lambda^n d_n} . \tag{6.19}$$

Similar results can be obtained for the modified Fredholm approximation should the first trace t_1 be infinite.

7. Nonsymmetric Kernels

If $k(x,y)$ is not symmetric, then the operators K and $A = I + \lambda K$ are not self-adjoint. However, if A is still bounded below away from zero in the sense that

$$\|A\Phi\| \geq \gamma\|\Phi\|, \quad \gamma > 0, \quad \text{for all } \Phi \in \mathcal{H} , \tag{7.1}$$

the bivariational bounds

$$J(\Psi,\Phi) - \gamma^{-1}\|A\Phi-f\| \, \|A^*\Psi-g\| \leq \langle g,\phi\rangle \leq J(\Psi,\Phi) + \gamma^{-1}\|A\Phi-f\| \, \|A^*\Psi-g\| \tag{7.2}$$

can be derived, $A^* = I + \lambda K^*$ being the adjoint of A. When $A = A^*$, these bounds are coarser than those in (3.13). Whenever (7.1) holds, it is possible to infer pointwise bivariational bounds on $\phi(x')$ by taking $g(x) = k(x',x)$, although the special role played by equation (2.7) specifying the reciprocal kernel is lost. In (7.2), $\Psi(x,x')$ would not now approximate the reciprocal kernel of $k(x,x')$, but rather that of $k(x',x)$. Details are given elsewhere [9].

Acknowledgement

I am most grateful to Dr. Louis Rall for helpful discussions about this work.

REFERENCES

[1] A. T. Lonseth, 1954, Bulletin of the American Mathematical Society, 60, 415-430.

[2] M. F. Barnsley and P. D. Robinson, 1974, Proc. Roy. Soc. A338, 527-533.

[3] R. J. Cole and D. C. Pack, 1975, Proc. Roy. Soc. A347, 239-252.

[4] J. S. R. Chisholm, 1970, 'Padé approximants and linear integral equations' in The Padé approximant in Theoretical Physics, ed. G. A. Baker and J. L. Gamel (Academic Press).

[5] M. F. Barnsley, 1975, J. Math. Phys. 4, 918-928.

[6] M. F. Barnsley and P. D. Robinson, 1974, J. Inst. Math. Applics 14, 229-249.

[7] F. Smithies, 1962, Integral Equations (Cambridge University Press).

[8] J. A. Cochran, 1972, The analysis of linear integral equations (McGraw-Hill).

[9] M. F. Barnsley and P. D. Robinson, 1976, Proc. Roy. Soc., Edinb., 75A, 109-118.

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
1767

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
POINTWISE BIVARIATIONAL BOUNDS ON SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Peter D. Robinson

8. CONTRACT OR GRANT NUMBER(s)
DAAG29-75-C-0024

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Work Unit Number 1 - Applied Analysis

11. CONTROLLING OFFICE NAME AND ADDRESS
U. S. Army Research Office
P.O. Box 12211
Research Triangle Park, North Carolina 27709

12. REPORT DATE
July 1977

13. NUMBER OF PAGES
13

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Integral equation, Symmetric kernel, Pointwise bivariational bounds, Hilbert space, Neumann, Padé, Fredholm

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Pointwise bounds of a bivariational nature are derived on the solution of a standard Fredholm integral equation of the second kind, with a symmetric kernel. The bounding functionals involve two trial vectors, one approximating the solution and the other approximating the reciprocal kernel. Even with the latter taken as the null vector, there is significant improvement over a previous approach. It is shown how suitable choices of trial vector lead to expressions for bounds on Neumann, Padé and Fredholm approximations.